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On sufficiency and duality in multiobjective programming problem under generalized α-type I univexity

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Abstract In this paper, we are concerned with the multiobjective programming problem with inequality constraints. We introduce new classes of generalized α-univex type I vector valued functions. A number of Kuhn–Tucker type sufficient optimality conditions are obtained for a feasible solution to be an efficient solution. The Mond–Weir type duality results are also presented.

Keywords Generalized convexity · α-type I univexity · Multiobjective programming · Sufficient optimality conditions · Duality

1 Introduction

Multiobjective optimization is known as an useful mathematical model to investigate some real-world problems with conflicting objectives, arising from economics, human decision-making, optimization and control, engineering, transportation and many others. For more applications and historical reference, see [3,17].

The concept of convexity and its various generalizations is important for quantitative and qualitative studies in operations research or applied mathematics. But for many mathematical models used in decision sciences, economics, management sciences, stochastics, applied mathematics and engineering, the notion of convexity does not no longer suffice. In recent years, there has been an increasing interest in generalization of convexity in connection with sufficiency and duality in optimization problems. It has been found that only a few properties

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of convex functions are needed for establishing sufficiency and duality theorems. Using the properties needed as definitions of new classes of functions, it is possible to generalize the notion of convexity and to extend the validity of theorems to larger classes of optimization problems. Consequently, several classes of generalized convex functions are introduced in the literature.

Parallel to the above development in multiple-objective optimization, there has been a very popular growth and application on invexity theory which was originated by Hanson [6] but so named by Craven [4]. Later Hanson and Mond [7] introduced type I and type II invexities which have been further generalized to pseudo type I, and quasi type I functions by Reuda and Hanson [18] and pseudoquasi type I, quasi pseudo type I and strictly pseudo quasi type I functions by Kaul et al. [10]. Reuda et al. [19] obtained optimality and duality results for several mathematical programs by combining the concepts of type I functions and univex functions [2]. Mishra [12] obtained optimality, duality and saddle point results for a multiple-objective programs by combining the concepts of pseudoquasi type I, quasi-pseudo type I, strictly pseudoquasi type I and univex functions.

Recently, Aghezzaf and Hachimi [1] introduced generalized type I vector valued functions and established duality theorems for Mond–Weir and general Mond–Weir type duality. Mishra et. al. [13] introduced generalized univex type I vector valued functions by extending the definition of generalized type I vector-valued functions introduced by Aghezzaf and Hachimi [1] and established Kuhn–Tucker type sufficient optimality conditions and duality theorems for Mond–Weir and general Mond–Weir type duality.

It is known that, despite substituting invexity for convexity, many theoretical problems in differentiable programming can also be solved; see Hanson [6], Edugo and Hanson [5], and Jeyakumar and Mond [9]. Noor [16] introduced some classes of α-invex functions by relaxing the definition of an invex function and studied some properties of the α-preinvex functions and their differentials. Mishra et al. [14] introduced the concept of the strict pseudo α-invex and quasi α-invex functions. Recently, Jayswal [8] discussed the sufficient optimality conditions for a class of nondifferentiable minimax fractional programming problem and established weak and strong duality theorems for two types of dual problems under generalized α-univexity.

In this paper, we consider a nonlinear multiobjective programming problem with inequality constraints and introduce new classes of generalized α-univex type I function. In Sect. 2, we introduce some preliminaries. Some sufficient optimality results are established in Sect. 3. A number of duality theorems in the Mond–Weir setting [15] are shown in Sect. 4.

2 Preliminaries

The following convention for inequalities will be used throughout the paper. If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in R^n$, we denote

$$\begin{aligned} x \leq y &\Leftrightarrow x_i \leq y_i \quad \forall i = 1, 2, \dots, n \quad \text{and} \quad x \neq y; \\ x \leqq y &\Leftrightarrow x_i \leq y_i \quad \forall i = 1, 2, \dots, n; \\ x < y &\Leftrightarrow x_i < y_i \quad \forall i = 1, 2, \dots, n. \end{aligned}$$

Consider the following multiobjective optimization problem:

$$\text{(P)} \qquad \begin{aligned} &\min\ f(x) \\ &\text{subject to } g(x) \leqq 0, \quad x \in X, \end{aligned}$$

where $f : X \to R^p$, $g : X \to R^m$, X is a non empty open subset of R^n.

Let $D = \{x \in X : g(x) \leqq 0\}$ be the set of all feasible solution for (P) and denote $I = \{1, 2, \ldots, p\}$, $M = \{1, 2, \ldots, m\}$, $J(x) = \{j \in M : g_j(x) = 0\}$ and $\tilde{J}(x) = \{j \in M : g_j(x) < 0\}$. It is obvious that $J(x) \cup \tilde{J}(x) = M$.

Let X be a nonempty subset of R^n, $\eta : X \times X \to R^n$ is an n-dimensional vector valued function and $\alpha(x, u) : X \times X \to R_+ \backslash \{0\}$ be a bifunction. First, we recall the following definitions.

Definition 2.1 (Noor [16]) A subset X is said to be α-invex set, if there exist $\eta : X \times X \to R^n$, $\alpha(x, u) : X \times X \to R_+$ such that

$$u + \lambda\alpha(x, u)\eta(x, u) \in X \qquad \forall x, u \in X, \lambda \in [0, 1].$$

Note that α-invex set need not be a convex set, see Noor [16].

From now onward we assume that the set X is a nonempty α-invex set with respect to $\alpha(., .)$ and $\eta(., .)$ unless otherwise specified.

Definition 2.2 (Noor [16]) The function f on the α-invex set is said to be α-preinvex function, if there exist $\eta : X \times X \to R^n$, $\alpha(x, u) : X \times X \to R_+$ such that

$$f(u + \lambda\alpha(x, u)\eta(x, u)) \leq (1 - \lambda) f(u) + \lambda f(x) \quad \forall x, u \in X, \ \ \lambda \in [0, 1].$$

Definition 2.3 (Noor [16]) The function f is said to be α-invex at $u \in X$ with respect to α and η, if there exist functions α and η such that, for every $x \in X$, we have

$$f(x) - f(u) \geq \langle \alpha(x, u)\nabla f(u), \eta(x, u)\rangle.$$

In the following definitions, $b_0, b_1 : X \times X \times [0, 1] \to R^+$, $b(x, a) = \lim_{\lambda \to 0} b(x, a, \lambda) \geq 0$, and b does not depend on λ if the corresponding functions are differentiable, $\phi_0, \phi_1 : R \to R$ and $\eta : X \times X \to R^n$ is a vector valued function.

Definition 2.4 (f, g) is said to be weak strictly pseudoquasi-α-type I univex at $u \in X$ if there exist $b_0, b_1, \phi_0, \phi_1, \alpha$and η such that

$$b_0(x, u)\phi_0[f(x) - f(u)] \leq 0 \Rightarrow \langle \alpha(x, u)\nabla f(u), \eta(x, u)\rangle < 0,$$
$$-b_1(x, u)\phi_1[g(u)] \leqq 0 \Rightarrow \langle \alpha(x, u)\nabla g(u), \eta(x, u)\rangle \leqq 0.$$

Example 2.1 $f(x) = (x_1 e^{\sin x_2}, x_2(x_2 - 1)e^{\cos x_1})$, $g(x) = (2x_1 + x_2 - 2)$ are weak strictly pseudoquasi α-type I univex at $u = (0, 0)$ with respect to $b_0(x, u) = b_1(x, u) = 1, \alpha(x, u) = 1, \eta(x, u) = (x_1 + x_2 - 1, x_2 - x_1)$ and ϕ_0, ϕ_1 are identity function on R.

Definition 2.5 (f, g) is said to be strong pseudoquasi-α-type I univex at $u \in X$ if there exist $b_0, b_1,\phi_0, \phi_1, \alpha$ and η such that

$$b_0(x, u)\phi_0[f(x) - f(u)] \leq 0 \Rightarrow \langle \alpha(x, u)\nabla f(u), \eta(x, u)\rangle \leq 0,$$
$$-b_1(x, u)\phi_1[g(u)] \leqq 0 \Rightarrow \langle \alpha(x, u)\nabla g(u), \eta(x, u)\rangle \leqq 0.$$

Example 2.2 $f(x) = (x_1(x_1 - 1)^2, x_2(x_2 - 1)^2(x_2^2 + 2))$, $g(x) = (x_1^2 + x_2^2 - 9)$ are strong pseudoquasi α-type I univex at $u = (0, 0)$ with respect to $b_0(x, u) = b_1(x, u) = 1$ $\alpha(x, u) = 1, \eta(x, u) = (x_1 - 1, x_2 - 1)$ and ϕ_0, ϕ_1 are identity function on R, but (f, g) is not weak strictly pseudoquasi α-type I univex with respect to same b_0, b_1, α and η as can be seen by taking $x = (1, -1)$.

Definition 2.6 (f, g) is said to be weak quasi strictly pseudo-α-type I univex at $u \in X$ if there exist $b_0, b_1, \phi_0, \phi_1, \alpha$ and η such that

$$b_0(x,u)\phi_0[f(x)-f(u)] \leq 0 \Rightarrow \langle \alpha(x,u)\nabla f(u), \eta(x,u)\rangle \leqq 0,$$
$$-b_1(x,u)\phi_1[g(u)] \leqq 0 \Rightarrow \langle \alpha(x,u)\nabla g(u), \eta(x,u)\rangle \leq 0.$$

Example 2.3 $f(x) = \left(x_1^3\left(x_1^2+1\right), x_2^2(x_2-1)^3\right)$, $g(x) = \left((2x_1-4)\mathrm{e}^{-x_2^2}, (x_1+x_2-2)(x_1^2+2x_1+4)\right)$ are weak quasi strictly pseudo-α-type I univex at $u = (0,0)$ with respect to $b_0(x,u) = b_1(x,u) = 1$, $\alpha(x,u) = 1$, $\eta(x,u) = (x_1, x_2(1-x_2))$ and ϕ_0, ϕ_1 are identity function on R.

Definition 2.7 (f, g) is said to be weak strictly pseudo-α-type I univex at $u \in X$ if there exist $b_0, b_1, \phi_0, \phi_1, \alpha$ and η such that

$$b_0(x,u)\phi_0[f(x)-f(u)] \leq 0 \Rightarrow \langle \alpha(x,u)\nabla f(u), \eta(x,u)\rangle < 0,$$
$$-b_1(x,u)\phi_1[g(u)] \leqq 0 \Rightarrow \langle \alpha(x,u)\nabla g(u), \eta(x,u)\rangle < 0.$$

Definition 2.8 A point $\bar{x} \in D$ is said to be an efficient solution for (P) if there exist no $x \in D$ such that $f(x) \leq f(\bar{x})$.

3 Sufficient optimality condition

In this section, we establish some sufficient optimality conditions under various generalized α-type I univex functions defined in the previous section.

Theorem 3.1 (Sufficiency). *Suppose that*

(i) $\bar{x} \in D$;
(ii) *there exist* $\bar{u} \in R^p, \bar{u} > 0, \bar{v} \in R^m$ *and* $\bar{v} \leqq 0$ *such that*
   (a) $\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0,$
   (b) $\bar{v}g(\bar{x}) = 0,$
   (c) $\bar{u}e = 1$, *where* $e = (1, 1, \ldots, 1)^T \in R^p$;
(iii) $(f, \bar{v}g)$ *is strong pseudoquasi* α*-type I univex at* $\bar{x} \in D$ *with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha$ *and* η.

Further assume that $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ *and* $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$.
Then $\bar{x}$ *is an efficient solution for (P).*

Proof Suppose contrary to the result that $\bar{x}$ is not an efficient solution to (P). Then there exists a feasible solution x to (P) such that

$$f_i(x) \leq f_i(\bar{x}) \quad \text{for any} \quad i \in \{1, 2, \ldots, p\}.$$

Since $b_0 \geq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we have

$$b_0(x,\bar{x})\phi_0[f_i(x)-f_i(\bar{x})] \leq 0 \quad \text{for any} \quad i \in \{1, 2, \ldots, p\}. \tag{1}$$

By the feasibility of $\bar{x}$, we have

$$-\bar{v}g(\bar{x}) \leqq 0.$$

Since $b_1 \geqq 0$ and $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we have

$$-b_1(x,\bar{x})\phi_1[\bar{v}g(\bar{x})] \leqq 0. \tag{2}$$

From (1), (2) and condition (iii), we have

$$\langle \alpha(x,\bar{x})\nabla f(\bar{x}), \eta(x,\bar{x})\rangle \leq 0,$$

and

$$\langle \alpha(x,\bar{x})\bar{v}\nabla g(\bar{x}), \eta(x,\bar{x})\rangle \leqq 0.$$

Since $\bar{u} > 0$, the above inequalities give

$$\langle \alpha(x,\bar{x})(\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x})), \eta(x,\bar{x})\rangle < 0.$$

Since $\alpha(x,\bar{x}) > 0$, from the above inequality, we have

$$\langle \bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}), \eta(x,\bar{x})\rangle < 0,$$

which contradicts (a) because $\langle 0, x\rangle = 0$. This completes the proof. □

Theorem 3.2 (Sufficiency). *Suppose that*

(i) $\bar{x} \in D$;
(ii) *there exist* $\bar{u} \in R^p$, $\bar{u} \geq 0$, $\bar{v} \in R^m$ *and* $\bar{v} \geq 0$ *such that*

  (a) $\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0$,
  (b) $\bar{v}g(\bar{x}) = 0$,
  (c) $\bar{u}e = 1$, *where* $e = (1, 1, \ldots, 1) \in R^p$;

(iii) $(f, \bar{v}g)$ *is weak strictly pseudoquasi* α*-type I univex at* $\bar{x} \in D$ *with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha$ *and* η.

Further assume that $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ *and* $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$. *Then* $\bar{x}$ *is an efficient solution for* (P).

Proof Suppose contrary to the result that $\bar{x}$ is not an efficient solution to (P). Then there exists a feasible solution x to (P) such that

$$f_i(x) \leq f_i(\bar{x}) \quad \text{for any} \quad i \in \{1, 2, \ldots, p\}.$$

Since $b_0 \geq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we have

$$b_0(x,\bar{x})\phi_0[f_i(x) - f_i(\bar{x})] \leq 0 \quad \text{for any} \quad i \in \{1, 2, \ldots, p\}. \tag{3}$$

By the feasibility of $\bar{x}$, we have

$$-\bar{v}g(\bar{x}) \leqq 0.$$

Since $b_1 \geqq 0$ and $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we have

$$-b_1(x,\bar{x})\phi_1[\bar{v}g(\bar{x})] \leqq 0. \tag{4}$$

From (3), (4) and condition (iii), we have

$$\langle \alpha(x,\bar{x})\nabla f(\bar{x}), \eta(x,\bar{x})\rangle < 0,$$

and

$$\langle \alpha(x,\bar{x})\bar{v}\nabla g(\bar{x}), \eta(x,\bar{x})\rangle \leqq 0.$$

Since $\bar{u} \geq 0$, the above inequalities give

$$\langle \alpha(x, \bar{x})(\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x})), \eta(x, \bar{x})\rangle < 0.$$

Since $\alpha(x, \bar{x}) > 0$, from the above inequality, we have

$$\langle \bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}), \eta(x, \bar{x})\rangle < 0,$$

which contradicts (a) because $\langle 0, x\rangle = 0$. This completes the proof. □

Theorem 3.3 (Sufficiency). *Suppose that*

(i) $\bar{x} \in D$;
(ii) *there exist* $\bar{u} \in R^p, \bar{u} \geqq 0, \bar{v} \in R^m$ *and* $\bar{v} \geqq 0$ *such that*

(a) $\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0$,
(b) $\bar{v}g(\bar{x}) = 0$,
(c) $\bar{u}e = 1$, *where* $e = (1, 1, \ldots, 1) \in R^p$;

(iii) $(f, \bar{v}g)$ *is weak strictly pseudo* α*-type I univex at* $\bar{x} \in D$ *with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha$ *and* η.

Further assume that $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ *and* $a \leq 0 \Rightarrow \phi_1(a) \leqq 0$.
Then $\bar{x}$ *is an efficient solution for (P).*

Proof Suppose contrary to the result that $\bar{x}$ is not an efficient solution to (P). Then there exists a feasible solution x to (P) such that

$$f_i(x) \leq f_i(\bar{x}) \quad \text{for any} \quad i \in \{1, 2, \ldots, p\}.$$

Since $b_0 \geq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we have

$$b_0(x, \bar{x})\phi_0[f_i(x) - f_i(\bar{x})] \leq 0 \quad \text{for any} \quad i \in \{1, 2, \ldots, p\}. \tag{5}$$

By the feasibility of $\bar{x}$, we have

$$-\bar{v}g(\bar{x}) \leqq 0.$$

Since $b_1 \geqq 0$ and $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we have

$$-b_1(x, \bar{x})\phi_1[\bar{v}g(\bar{x})] \leqq 0. \tag{6}$$

From (5), (6) and condition (iii), we have

$$\langle \alpha(x, \bar{x})\nabla f(\bar{x}), \eta(x, \bar{x})\rangle < 0,$$

and

$$\langle \alpha(x, \bar{x})\bar{v}\nabla g(\bar{x}), \eta(x, \bar{x})\rangle < 0.$$

Since $\bar{u} \geqq 0$, the above inequalities give

$$\langle \alpha(x, \bar{x})(\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x})), \eta(x, \bar{x})\rangle < 0.$$

Since $\alpha(x, \bar{x}) > 0$, from the above inequality, we have

$$\langle \bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}), \eta(x, \bar{x})\rangle < 0,$$

which contradicts (a) because $\langle 0, x\rangle = 0$. This completes the proof. □

4 Mond–Weir duality

Now, in relation to (P) we consider the following dual problem, which is in the format of Mond–Weir [15]:

$$\text{(MWD)} \quad \text{maximize } f(y) \tag{7}$$
$$\text{subject to } u\nabla f(y) + v\nabla g(y) = 0,$$
$$vg(y) \geq 0, \tag{8}$$
$$v \geq 0, u \geq 0 \text{ and } ue = 1, \tag{9}$$

where $e = (1, 1, \ldots, 1)^T \in R^p$.

Let $W = \{(y, u, v) : u\nabla f(y) + v\nabla g(y) = 0, vg(y) \geqq 0, u \in R^p, v \in R^m, v \geqq 0\}$ denote the set of all feasible solutions of (MWD).

Theorem 4.1 (Weak duality). *Suppose that*

(i) $x \in D$;
(ii) $(y, u, v) \in W$ *and* $u > 0$;
(iii) (f, vg) *is strong pseudoquasi* α*-type I invex at* y *with respect to* $b_0, b_1\phi_0, \phi_1, \alpha$ *and* η.

Further assume that $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ *and* $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$.

Then the following can not hold:

$$f(x) \leq f(y).$$

Proof Suppose contrary to the result, i.e.,

$$f_i(x) \leq f_i(y) \quad \text{for all} \quad i = \{1, 2, \ldots, p\}.$$

Since $b_0 \geq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we have

$$b_0(x, y)\phi_0[f_i(x) - f_i(y)] \leq 0 \quad \text{for all} \quad i = \{1, 2, \ldots, p\}. \tag{10}$$

By the feasibility of (y, u, v), we have

$$-vg(y) \leqq 0.$$

Since $b_1 \geqq 0$ and $a \leq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we have

$$-b_1(x, y)\phi_1[vg(y)] \leqq 0. \tag{11}$$

From (10), (11) and condition (iii), we have

$$\langle \alpha(x, y)\nabla f(y), \eta(x, y)\rangle \leq 0,$$

and

$$\langle \alpha(x, y)v\nabla g(y), \eta(x, y)\rangle \leqq 0.$$

Since $u > 0$, the above inequalities give

$$\langle \alpha(x, y)(u\nabla f(y) + v\nabla g(y)), \eta(x, y)\rangle < 0.$$

Since $\alpha(x, y) > 0$, from the above inequality, we have

$$\langle u\nabla f(y) + v\nabla g(y), \eta(x, y)\rangle < 0,$$

which contradicts (7) because $\langle 0, x\rangle = 0$. This completes the proof. □

Theorem 4.2 (Weak duality). *Suppose that*

(i) $x \in D$;
(ii) $(y, u, v) \in W$ *and* $u \geq 0$;
(iii) (f, vg) *is weak strictly pseudoquasi* α*-type I univex at* y *with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha$ *and* η.

Further assume that $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ *and* $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$.
Then the following can not hold:

$$f(x) \leq f(y).$$

Proof Suppose contrary to the result, i.e.,

$$f_i(x) \leq f_i(y) \quad \text{for all} \quad i = \{1, 2, \ldots, p\}.$$

Since $b_0 \geq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we have

$$b_0(x, y)\phi_0[f_i(x) - f_i(y)] \leq 0 \quad \text{for all} \quad i = \{1, 2, \ldots, p\}. \tag{12}$$

By the feasibility of (y, u, v), we have

$$-vg(y) \leqq 0.$$

Since $b_1 \geqq 0$ and $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we have

$$-b_1(x, y)\phi_1[vg(y)] \leqq 0. \tag{13}$$

From (12), (13) and condition (iii), we have

$$\langle \alpha(x, y)\nabla f(y), \eta(x, y)\rangle < 0,$$

and

$$\langle \alpha(x, y)v\nabla g(y), \eta(x, y)\rangle \leqq 0.$$

Since $u \geq 0$, the above inequalities give

$$\langle \alpha(x, y)(u\nabla f(y) + v\nabla g(y)), \eta(x, y)\rangle < 0.$$

Since $\alpha(x, y) > 0$, from the above inequality, we have

$$\langle u\nabla f(y) + v\nabla g(y), \eta(x, y)\rangle < 0,$$

which contradicts (7) because $\langle 0, x\rangle = 0$.This completes the proof. □

Theorem 4.3 (Weak duality). *Suppose that*

(i) $x \in D$;
(ii) $(y, u, v) \in W$;
(iii) (f, vg) *is weak strictly pseudo* α*-type I univex at* y *with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha$ *and* η.

Further assume that $a \leq 0 \Rightarrow \phi_0(a) \leq 0$ *and* $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$.
Then the following can not hold:

$$f(x) \leq f(y).$$

Proof Suppose contrary to the result, i.e.,

$$f_i(x) \leq f_i(y) \quad \text{for all} \quad i = \{1, 2, \ldots, p\}.$$

Since $b_0 \geq 0$ and $a \leq 0 \Rightarrow \phi_0(a) \leq 0$, from the above inequality, we have

$$b_0(x, y)\phi_0[f_i(x) - f_i(y)] \leq 0 \quad \text{for all} \quad i = \{1, 2, \ldots, p\}. \tag{14}$$

By the feasibility of (y, u, v), we have

$$-vg(y) \leqq 0.$$

Since $b_1 \geqq 0$ and $a \leqq 0 \Rightarrow \phi_1(a) \leqq 0$, from the above inequality, we have

$$-\, b_1(x, y)\phi_1[vg(y)] \leqq 0. \tag{15}$$

From (14), (15) and condition (iii), we have

$$\langle \alpha(x, y)\nabla f(y), \eta(x, y)\rangle < 0,$$

and

$$\langle \alpha(x, y)v\nabla g(y), \eta(x, y)\rangle < 0.$$

Since $u \geqq 0$, the above inequalities give

$$\langle \alpha(x, y)(u\nabla f(y) + v\nabla g(y)), \eta(x, y)\rangle < 0.$$

Since $\alpha(x, y) > 0$, from the above inequality, we have

$$\langle u\nabla f(y) + v\nabla g(y), \eta(x, y)\rangle < 0,$$

which contradicts (7) because $\langle 0, x\rangle = 0$. This completes the proof. □

Theorem 4.4 (Strong duality). *Let $\bar{x}$ be an efficient solution for (P) and $\bar{x}$ satisfies a constraints qualification for (P) in Marusciac* [11]. *Then there exists $\bar{u} \in R^p$ and $\bar{v} \in R^m$ such that $(\bar{x}, \bar{u}, \bar{v})$ is feasible for (MWD). If any of the weak duality in Theorems* 4.1–4.3 *also holds, then $(\bar{x}, \bar{u}, \bar{v})$ is an efficient solution for (MWD).*

Proof Since $\bar{x}$ is efficient for (P) and satisfy the constraints qualification for (P), then from Kuhn–Tucker necessary optimality condition, we obtain $\bar{u} > 0$ and $\bar{v} \geqq 0$ such that

$$\bar{u}\nabla f(\bar{x}) + \bar{v}\nabla g(\bar{x}) = 0, \quad \bar{v}g(\bar{x}) = 0.$$

The vector $\bar{u}$ may be normalized according to $\bar{u}e = 1, \bar{u} > 0$, which gives that the triplet $(\bar{x}, \bar{u}, \bar{v})$ is feasible for (MWD). The efficiency of $(\bar{x}, \bar{u}, \bar{v})$ for (MWD) follows from weak duality theorem. This completes the proof. □

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